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Approach to Equilibrium in a One-Dimensional, Two-Component Gas of Maxwellian Molecules

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The coupled Boltzmann equations describing the evolution of the velocity distributions of a one-dimensional, two-component gas of Maxwellian molecules are analyzed. When the two species have different masses, the system approaches equilibrium. The complete eigenvalue spectrum of the linearized collision operator is obtained, and is found to exhibit an interesting dependence on the mass ratio. The response of one species to an external field, when the other species is regarded as a host fluid, is also examined.

KEY WORDS: Boltzmann equation; relaxation time; two-component gas.

1. INTRODUCTION

The kinetic theory of one-dimensional gas models has been of interest for some time. For the hard-rod system, exact solutions for the dynamics have been obtained.[1-3] Subsequently, Resibois[4] solved the problem of approach to equilibrium of a hard-rod gas immersed in a host fluid having a stationary velocity distribution, and Piasecki[5,6] examined the response of this system to an external field. A related model, consisting of a mixture of hard rods with two different masses, has been studied by Marro and Masoliver.[7,8] This model has the virtue of intrinsic approach to equilibrium (that is, without the assistance of a host fluid maintained in equilibrium). In a recent simulation,[8] the dependence of the relaxation time on mass ratio was examined.

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This paper is concerned with the kinetic theory of a one-dimensional, two-component gas of "soft" (Maxwellian) point particles, at the level of the Boltzmann equation. The eigenvalue spectrum of the linearized collision operator is obtained in Section 2, by an elementary method. The dependence of the relaxation time on mass ratio is found to be qualitatively similar to that observed in Ref. 8. In Section 3 the model is modified, with the massive particles regarded as a host fluid, and the response of the light particles to an external field is examined.

2. ONE-DIMENSIONAL GAS MIXTURE

Consider a one-dimensional gas of point particles, composed of two species, one with unit mass, the other with mass M. In like-species collisions the particles simply exchange velocities, and so these events may be ignored in studying the relaxation of the velocity distribution. Momentum and energy exchange in interspecies collisions *do* lead to relaxation.[7] In this section we examine the Boltzmann equation which describes this process.

If u and v are the velocities of a mass-M and unit mass particle, respectively, prior to collision, then after they collide their velocities are

$$u' = \frac{2v + (M-1)\,u}{M+1}, \qquad v' = \frac{2Mu - (M-1)\,v}{M+1} \tag{1}$$

Let $\phi(v, t)$ and $\psi(v, t)$ be normalized velocity distributions for the mass-M and unit mass species, respectively. The evolution of these distributions is governed by the coupled Boltzmann equations

$$\frac{\partial\psi(v, t)}{\partial t} = n_M \int du\, |u-v|\, \sigma(|u-v|)[\psi(v', t)\,\phi(u', t) - \psi(v, t)\,\phi(u, t)] \tag{2}$$

$$\frac{\partial\phi(v, t)}{\partial t} = n_1 \int du\, |u-v|\, \sigma(|u-v|)[\psi(u', t)\,\phi(v', t) - \psi(u, t)\,\phi(v, t)] \tag{3}$$

where n_M and n_1 are the number densities of the species, and we have assumed spatial uniformity and the absence of any external force acting on the particles. With the further assumption that the collision cross section is inversely proportional to relative velocity,

$$\sigma(|u|) = \frac{\sigma_0}{|u|} \tag{4}$$

i.e., "Maxwellian molecules,"determination of the eigenvalue spectrum of the linearized collision operator becomes quite straightforward. Accordingly, we linearize ψ and ϕ about equilibrium

$$\psi(v, t) = \psi_{\text{eq}}(v)[1 + \psi'(v, t)] \tag{5}$$

$$\phi(v, t) = \phi_{\text{eq}}(v)[1 + \phi'(v, t)] \tag{6}$$

where

$$\phi_{\text{eq}}(v) = \left(\frac{M}{2\pi k_B T}\right)^{1/2} e^{-Mv^2}/2kT \tag{7}$$

and similarly for $\psi_{\text{eq}}(M \to 1)$. In what follows, we take $n_1 = n_M = n$. Let

$$x = u/(k_B T)^{1/2} \tag{8}$$

$$y = v/(k_B T)^{1/2} \tag{9}$$

$$\tau = \frac{n\sigma_0}{(2\pi)^{1/2}} t \tag{10}$$

and let

$$\bar{\psi}(y, \tau) = \psi'\left((k_B T)^{1/2} y, \frac{(2\pi)^{1/2}}{n\sigma_0} \tau\right) \tag{11}$$

$$\bar{\phi}(y, \tau) = \phi'\left((k_B T)^{1/2} y, \frac{(2\pi)^{1/2}}{n\sigma_0} \tau\right) \tag{12}$$

Using Eqs. (4)–(12), the Boltzmann equations, linearized about equilibrium, may be expressed in dimensionless form

$$\begin{aligned}\frac{\partial}{\partial\tau} \bar{\psi}(x, \tau) &= \sqrt{M} \int dy\, e^{-My^2/2}[\bar{\psi}(ay + bx, \tau) + \bar{\phi}(cy + dx, \tau) \\ &\quad - \bar{\psi}(x, \tau) - \bar{\phi}(y, \tau)] \\ &\equiv \hat{F}\bar{\psi}(x, \tau) + \hat{G}\bar{\phi}(x, \tau)\end{aligned} \tag{13a}$$

$$\begin{aligned}\frac{\partial}{\partial\tau} \bar{\phi}(x, \tau) &= \int dy\, e^{-y^2/2}[\bar{\psi}(ax + by, \tau) + \bar{\phi}(cx + dy, \tau) \\ &\quad - \bar{\psi}(y, \tau) - \bar{\phi}(x, \tau)] \\ &\equiv \hat{H}\bar{\psi}(x, \tau) + \hat{J}\bar{\phi}(x, \tau)\end{aligned} \tag{13b}$$

where, from Eqs. (1), (8), and (9) we have

$$d=\frac{2}{M+1}=\frac{a}{M} \tag{14a}$$

$$c=\frac{M-1}{M+1}=-b \tag{14b}$$

If we let $\Psi(x,\tau)$ stand for the vector $\binom{\psi(x,\tau)}{\phi(x,\tau)}$ then Eqs. (13a) and (13b) may be written

$$\left(\frac{\partial}{\partial\tau}+(2\pi)^{1/2}\right)\Psi=\hat{\mathscr{F}}\Psi \tag{15}$$

where

$$\hat{\mathscr{F}}=\begin{pmatrix}\hat{F}+(2\pi)^{1/2}\hat{I} & \hat{G}\\ \hat{H} & \hat{J}+(2\pi)^{1/2}\hat{I}\end{pmatrix} \tag{16}$$

with $\hat{I}$ the identity operator. Consider the Hilbert spaces $\mathscr{H}_i=L^2(\mathbb{R}, d\mu_i)$ where $\mu_1=e^{-x^2}/2$, $\mu_2=\sqrt{M}\,e^{-Mx^2/2}$. It is straightforward to show that $\hat{\mathscr{F}}$ is a self-adjoint, Hilbert–Schmidt integral operator on the direct sum $\mathscr{H}_1\oplus\mathscr{H}_2$. The spectrum of $\hat{\mathscr{F}}-(2\pi)^{1/2}\hat{\mathscr{I}}$ ($\hat{\mathscr{I}}$ being the identity matrix) is therefore bounded above and below, and may possess an accumulation point at $-(2\pi)^{1/2}$. We now introduce a basis for $\mathscr{H}_1\oplus\mathscr{H}_2$. Let

$$K_p(x)=\frac{1}{2^{p/2}\sqrt{p!}}H_p\left(\frac{x}{\sqrt{2}}\right) \tag{17a}$$

$$N_p(x)=\frac{1}{2^{p/2}\sqrt{p!}}H_p\left(\left(\frac{M}{2}\right)^{1/2}x\right) \tag{17b}$$

where H_p is the Hermite polynomial. These functions have the orthogonality properties

$$\int_{-\infty}^{\infty}dx\,e^{-x^2/2}K_p(x)\,K_{p'}(x)=(2\pi)^{1/2}\,\delta_{pp'} \tag{18a}$$

$$\int_{-\infty}^{\infty}dx\,e^{-Mx^2/2}N_p(x)\,N_{p'}(x)=\left(\frac{2\pi}{M}\right)^{1/2}\delta_{pp'} \tag{18b}$$

and their generating functions are

$$e^{s(x-s/2)}=\sum_{j=0}^{\infty}\frac{s^j}{\sqrt{j!}}K_j(x) \tag{19a}$$

$$e^{s(\sqrt{M}x-s/2)}=\sum_{j=0}^{\infty}\frac{s^j}{\sqrt{j!}}N_j(x) \tag{19b}$$

We expand the velocity distributions as follows:

$$\Psi(x, \tau) = \sum_{i,j} \begin{pmatrix} v_i(\tau)\, K_i(x) \\ \omega_j(\tau)\, N_j(x) \end{pmatrix} \tag{20}$$

where

$$\omega_j(\tau) = \left(\frac{M}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, e^{-Mx^2/2} N_j(x)\, \bar{\phi}(x, \tau) \tag{21}$$

and similarly for $v_j(\tau)$ if we take $M \to 1$, $N \to K$, and $\bar{\phi} \to \bar{\psi}$. Define the matrix elements

$$F_{j,k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2/2} K_j(x)\, \hat{F} K_k(x) \tag{22a}$$

$$G_{j,k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2/2} K_j(x)\, \hat{G} N_k(x) \tag{22b}$$

$$H_{j,k} = \left(\frac{M}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, e^{-Mx^2/2} N_j(x)\, \hat{H} K_k(x) \tag{22c}$$

$$J_{j,k} = \left(\frac{M}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, e^{-Mx^2/2} N_j(x)\, \hat{J} N_k(x) \tag{22d}$$

The matrix elements are readily computed using the generating functions. From Eqs. (13), (19a), and (22a) we have

$$\begin{aligned} \sum_{j,k} F_{j,k} \frac{s^j t^k}{\sqrt{j!\, k!}} &= \left(\frac{M}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx\, e^{-x^2/2 + s(x - s/2)} \\ &\quad \times \left[\int dy\, e^{-My^2/2} \left(e^{t(ay + bx - t/2)} - e^{t(x - t/2)}\right) \right] \\ &= (2\pi)^{1/2} (e^{bst} - e^{st}) \end{aligned} \tag{23}$$

so that

$$F_{j,k} = (2\pi)^{1/2} (b^k - 1)\, \delta_{j,k} \tag{24a}$$

In the same manner, one finds

$$H_{j,k} = G_{j,k} = (2\pi)^{1/2} (M^{k/2} d^k - \delta_{k,0})\, \delta_{j,k} \tag{24b}$$

$$J_{j,k} = (2\pi)^{1/2} (c^k - 1)\, \delta_{j,k} \tag{24c}$$

Thus, if we let $\mathbf{X} = (v_0, \omega_0, v_1, \omega_1, v_2, \omega_2, ...)$ then Eq. (13) is equivalent to

$$\frac{d\mathbf{X}}{d\tau} = D\mathbf{X} \tag{25}$$

with D block diagonal, the kth block having elements

$$\begin{pmatrix} F_{j,k} & G_{j,k} \\ G_{j,k} & J_{j,k} \end{pmatrix} \tag{26}$$

The eigenvalues of the linearized collision operator are those of each block on the diagonal of D. Since all (0, 0) elements vanish, there are two zero eigenvalues associated with the first block. The eigenvalues deriving from the kth block ($k \geqslant 1$) are

$$\lambda_k^{(\pm)} = \begin{cases} -(2\pi)^{1/2} \left[1 - \left(\dfrac{M-1}{M+1} \right)^k \mp \dfrac{2^k M^{k/2}}{(M+1)^k} \right], & k \text{ even} \\ -(2\pi)^{1/2} \left\{ 1 \mp \dfrac{[(M-1)^{2k} + 2^{2k} M^k]^{1/2}}{(M+1)^k} \right\}, & k \text{ odd} \end{cases} \tag{27}$$

The associated eigenvectors, $\mathbf{X}_k^{(\pm)}$, have $\omega_j = v_j = 0$, $j \neq k$, and

$$\begin{pmatrix} v_k \\ \omega_k \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, & k \text{ even} \\ \begin{pmatrix} 1 \\ -\dfrac{(M-1)^k}{2^k M^{k/2}} \pm \left[\dfrac{(M-1)^{2k}}{2^{2k} M^k} + 1 \right]^{1/2} \end{pmatrix}, & k \text{ odd} \end{cases} \tag{28}$$

From Eq. (27) we see that $\lambda_1^{(+)} = \lambda_2^{(+)} = 0$, and that the remaining eigenvalues are negative. The $\lambda = 0$ eigenmodes correspond to conservation of (i) the number of unit-mass particles:

$$\frac{1}{(2\pi)^{1/2}} \int dx\, e^{-x^2/2} \bar{\psi}(x, \tau) = v_0 = \frac{1}{2} (\mathbf{X}_0^{(+)} + \mathbf{X}_0^{(-)}) \cdot \mathbf{X} \tag{29}$$

(ii) the number of mass-M particles:

$$\left(\frac{M}{2\pi} \right)^{1/2} \int dx\, e^{-Mx^2/2} \bar{\phi}(x, \tau) = \omega_0 = \frac{1}{2} (\mathbf{X}_0^{(+)} - \mathbf{X}_0^{(-)}) \cdot \mathbf{X} \tag{30}$$

(iii) total momentum:

$$\left(\frac{k_B T}{2\pi} \right)^{1/2} \left[\int dx\, e^{-x^2/2} x \bar{\psi}(x, \tau) + M^{3/2} \int dx\, e^{-Mx^2/2} x \bar{\phi}(x, \tau) \right]$$

$$= (k_B T)^{1/2} (v_1 + \sqrt{M}\, \omega_1) = (k_B T)^{1/2}\, \mathbf{X}_1^{(+)} \cdot \mathbf{X} \tag{31}$$

(iv) total energy:

$$\frac{1}{2}\frac{k_B T}{\sqrt{2\pi}}\left[\int dx\, e^{-x^2/2} x^2 \bar{\psi}(x, \tau) + M^{3/2} \int dx\, e^{-Mx^2/2} x^2 \bar{\phi}(x, \tau)\right]$$
$$= \frac{k_B T}{2}[\sqrt{2}(v_2 + \omega_2) + v_0 + \omega_0] = \frac{k_B T}{2}(\sqrt{2}\,\mathbf{X}_2^{(+)} + \mathbf{X}_0^{(+)}) \cdot \mathbf{X} \quad (32)$$

These four conserved quantities correspond to the four degrees of freedom in the equilibrium state of a one-dimensional, two-component gas. They completely determine a time-independent velocity distribution.

The highest nonzero eigenvalue, which controls the long-time relaxation of the velocity distribution, is $\lambda_3^{(+)}$. For $M = 1$, and again when $M \to \infty$, $\lambda_3^{(+)} = 0$. In either of these extreme circumstances, the velocity distribution cannot relax to a Maxwellian, and the state variables—particle numbers, momentum, and energy—do not uniquely determine a time-independent distribution. For $M \gtrsim 1$, we have

$$\lambda_3^{(+)} = -\frac{3(2\pi)^{1/2}}{8}(M-1)^2 + O(M-1)^3 \quad (33)$$

while in the opposite limit, $M \gg 1$,

$$\lambda_3^{(+)} = \frac{6(2\pi)^{1/2}}{M} + O\left(\frac{1}{M^2}\right) \quad (34)$$

When $M = 3 + 2\sqrt{2} \simeq 5.8284$, $|\lambda_3^{(+)}|$ takes its largest value, $\lambda_3^{(+)} = (\pi/2)^{1/2}$. In this regard, it is interesting to note that their molecular dynamics simulation of a $1-d$, two-component gas of *hard* rods, Marro and Masoliver[8] found that the relaxation time was shortest for a mass ratio of about 5.

From Eq. (27) and Fig. 1, the following features of the spectrum are evident. When $M = 1$, all "+" eigenvalues are zero, and all "−" eigenvalues equal $-2(2\pi)^{1/2}$, the latter representing the mixing of two populations distinguished solely by their initial conditions. When $M \to \infty$, $\lambda_k^{(-)} = -2(2\pi)^{1/2}$ for k odd, and all other eigenvalues are zero. The nonzero eigenvalues in this case represent momentum relaxation of the distribution for the unit-mass particles, due to reflecting collisions.

For $1 < M < \infty$, there is a discrete spectrum bounded above by zero and below by $-2(2\pi)^{1/2}$, and with an accumulation point at $-(2\pi)^{1/2}$. For the special value $M = 3 + 2\sqrt{2}$, $|\lambda_k^{(+)}|$ takes its maximum, and $|\lambda_{2k+1}^{(-)}|$ its minimum value, while $\lambda_{2k}^{(-)} = -(2\pi)^{1/2}$. Also, $\lambda_{2k}^{(+)}$ and $\lambda_{2k-1}^{(+)}$ are degenerate at this point. For intermediate M values, the highest nonzero eigenvalues,

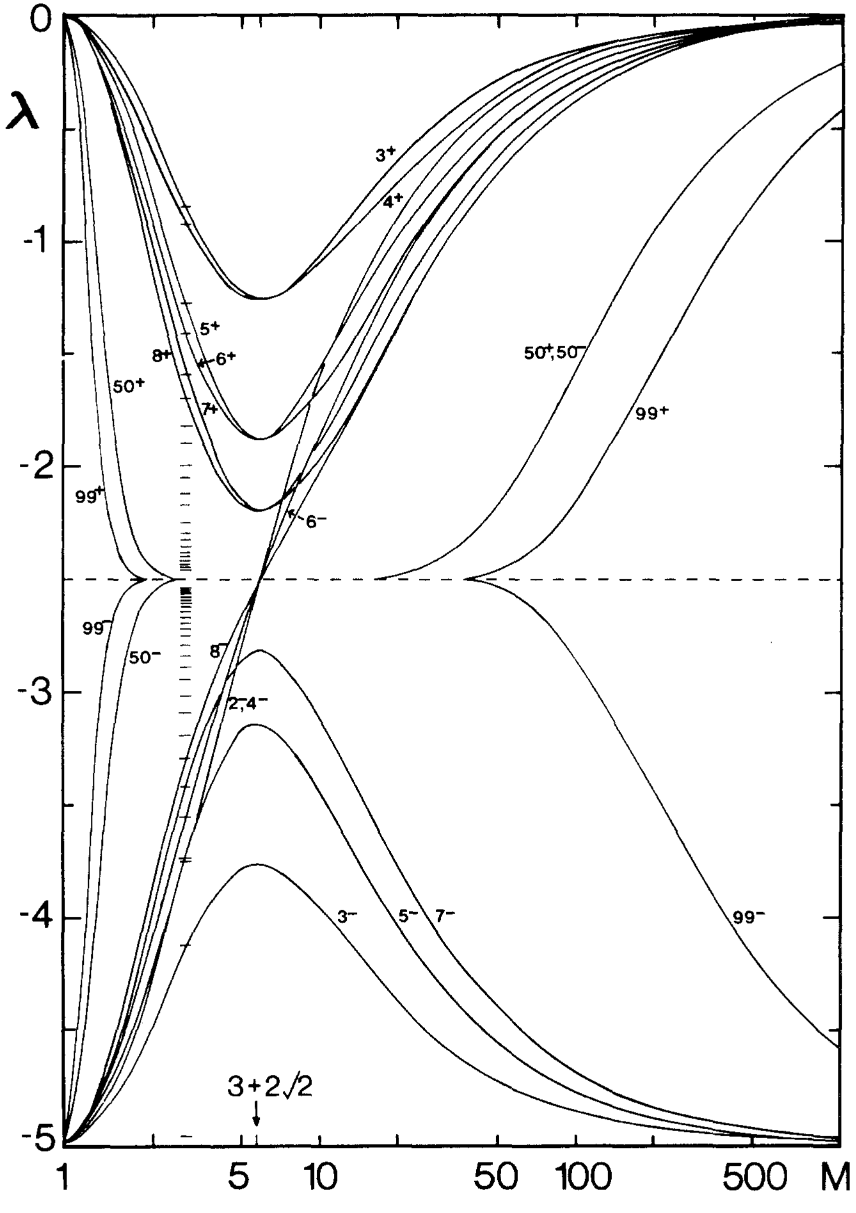

Fig. 1. Eigenvalue spectrum of the linearized collision operator, Eq. (34). Selected eigenvalues, labelled by number and sign, are plotted versus mass ratio M. The first 50 eigenvalues for $M = 3$ are denoted by horizontal lines.

$\lambda_3^{(+)}$ and $\lambda_4^{(+)}$, are nearly degenerate, but are well separated from the other eigenvalues. Deviations from equilibrium which are even in y, (i.e., energy, as opposed to momentum relaxation), have a relaxation time $1/|\lambda_4^{(+)}|$, which is in general slightly shorter than $1/|\lambda_3^{(+)}|$, which governs momentum relaxation.

As noted above, the spectrum of the collision operator is bounded below because this operator is the sum of a compact operator and a multiple of the identity. This is not true for the hard-core model, in which case we have, in place of Eq. (15),

$$\frac{\partial \Psi}{\partial \tau}(x, \tau) = \hat{\mathscr{J}}\,\Psi(x) - \hat{\mathscr{K}}(x)\,\Psi(x) \tag{35}$$

where $\hat{\mathscr{J}}$ is an integral operator and $\hat{\mathscr{K}}(x)$ is the *unbounded* multiplication operator

$$\hat{\mathscr{K}}(x) = (k_B T)^{1/2} \begin{pmatrix} (2\pi)^{1/2}\, x \operatorname{erf}\left[\left(\frac{M}{2}\right)^{1/2} x\right] + \frac{2}{\sqrt{M}}\, e^{-Mx^2/2} & 0 \\ 0 & (2\pi)^{1/2}\, x \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + 2e^{-x^2/2} \end{pmatrix} \tag{36}$$

Thus, in the case of hard-core particles, the collision operator spectrum is not bounded below. In the case of Maxwellian molecules in three dimensions, the spectrum is not bounded below, for the collision term includes a noncompact integral operator.[9]

3. GAS OF PARTICLES ACCELERATED BY AN EXTERNAL FIELD

To study the response to an external field, we modify the model considered in the preceding section in two respects: The unit-mass particles now experience a constant acceleration A, and the mass-M particles are taken as constituting a host fluid with stationary velocity distribution $\phi(v)$. The velocity distribution for the unit-mass particles is now governed by the linear Boltzmann equation

$$\frac{\partial \psi(v, t)}{\partial t} + A\,\frac{\partial \psi(v, t)}{\partial v} = n_M \int du\, |u - v|\, \sigma(|u - v|) \times [\psi(v', t)\,\phi(u') - \psi(v, t)\,\phi(u)] \tag{37}$$

The case of hard rods ($\sigma = 1$) interacting with a host fluid (also composed of unit-mass particles) at zero temperature $[\phi(v) = \delta(v)]$ was treated by

Piasecki,[5,6] who pointed out that in this case the response is inherently nonlinear: the mean velocity in the steady state is

$$\bar{v} = \left(\frac{2A}{\pi n_M}\right)^{1/2} \tag{38}$$

In what follows we consider Maxwellian molecules, with σ given by Eq. (4). Setting $M = 1$, (so that $v' = u$, and $u' = v$), Eq. (37) takes the form

$$\frac{\partial \psi(v, t)}{\partial t} + A \frac{\partial \psi(v, t)}{\partial v} = -\frac{1}{\bar{\tau}} [\psi(v, t) - \phi(v)] \tag{39}$$

where $\bar{\tau} = (n_M \sigma_0)^{-1}$.

It is not hard to show that for $\phi(v) = \delta(v)$, and $\psi(v, t=0) = \delta(v - v_0)$,

$$\psi(v, t) = e^{-(v - v_0)/A\bar{\tau}} \delta(v - v_0 - At) + (A\bar{\tau})^{-1} e^{-v/A\bar{\tau}} \theta(v)\, \theta(t - v/A) \tag{40}$$

where θ is the Heavyside step function, so that the average velocity is $\bar{v}(t) = v_0 e^{-t/\bar{\tau}} + A\bar{\tau}(1 - e^{-t/\bar{\tau}})$. From Eq. (39) it is also easily seen that for an arbitrary $\phi(v)$, normalized and with finite first moment, the average velocity in the steady state is

$$\bar{v} = A\bar{\tau} + \int_{-\infty}^{\infty} v\phi(v)\, dv \tag{41}$$

Hence for Maxwellian particles linear response holds quite generally.

Responses intermediate between Eq. (38) and Ohm's law may be obtained by taking $M = 1$, $\phi(v) = \delta(v)$, and $\sigma(|v|) = \sigma_0 |v|^{-\alpha}$. One then finds the steady state velocity distribution to be

$$\psi(v) = \text{const}\, \theta(v) \exp\left[-\frac{v^{2-\alpha}}{(2-\alpha) A\bar{\tau}}\right] \tag{42}$$

so that $\bar{v} \propto A^{1/(1-\alpha)}$. Evidently, the nature of the response to an external field is determined by the velocity dependence of the cross section for interspecies scattering.

Another case which is amenable to simple analysis is the "Lorentz" limit: infinitely massive, motionless, and uniformly distributed host-fluid particles. In the intervals between collisions, the unit-mass particles accelerate at a constant rate A, and at each collision they suffer a velocity reversal. Owing to the assumption of Maxwellian particles, the collision probability (per unit time) is independent of velocity. For this case Eq. (37) becomes

$$\frac{\partial \psi(v, t)}{\partial t} + A \frac{\partial \psi(v, t)}{\partial v} = -\frac{1}{\bar{\tau}} [\psi(v, t) - \psi(-v, t)] \tag{43}$$

Taking the first moment of this equation, one finds for the average velocity

$$\bar{v}(t) = \frac{A\bar{\tau}}{2}\,(1 - e^{-2t/\bar{\tau}}) \tag{44}$$

Although the average velocity saturates as $t \to \infty$, the velocity distribution does not attain a steady state. For if we define the moment-generating function

$$g(x, t) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,\overline{v^n(t)} \tag{45}$$

then from Eq. (43) we have

$$\frac{\partial g}{\partial t}(x, t) = Axg(x, t) - \frac{1}{\bar{\tau}}\,[g(x) - g(-x)] \tag{46}$$

If g is split into parts even and odd in x, then Eq. (46) yields a pair of simple coupled equations for the components of g. For the initial condition $\psi(v, 0) = \delta(v)$, which implies $g(x, 0) = 1$, one finds

$$\begin{aligned} g(x, t) = e^{-t/\bar{\tau}} \Bigg(&\cosh \left\{ [1 + (A\bar{\tau}x)^2]^{1/2} \frac{t}{\bar{\tau}} \right\} \\ &+ \frac{1 + A\bar{\tau}x}{[1 + (A\bar{\tau}x)^2]^{1/2}} \sinh \left\{ [1 + (A\bar{\tau}x)^2]^{1/2} \frac{t}{\bar{\tau}} \right\} \Bigg) \end{aligned} \tag{47}$$

which clearly does not approach a steady state. The mean-square velocity is

$$\overline{v^2(t)} = A^2\bar{\tau}\left[t + \frac{\bar{\tau}}{2}\,(e^{-2t/\bar{\tau}} - 1) \right] \tag{48}$$

For large t and small x

$$g(x, t) \simeq (1 + \tfrac{1}{2}\bar{\tau}Ax)\, e^{(1/2)\bar{\tau}A^2x^2t} \tag{49}$$

from which it is evident that $\overline{v^{2r}}$ and $\overline{v^{2r+1}}$ grow $\propto t^r$ for large t. In the case of particles *diffusing* in velocity space, and subject to constant acceleration A, the moment-generating function is

$$g_D(x, t) = e^{Axt} e^{Dx^2t} \tag{50}$$

where D is the diffusion coefficient and we have again assumed the initial condition $\psi(v, 0) = \delta(v)$. Comparing Eqs. (49) and (50), we see that acceleration and collisions combine to produce diffusion, while unbounded increase of the mean velocity is, as expected, thwarted by collisions.

For the intermediate case, $1 < M < \infty$, and with $\phi(v) = \delta(v)$ as before, Eq. (37) takes the form

$$\frac{\partial \psi(v, t)}{\partial t} + A \frac{\partial \psi(v, t)}{\partial v} = \frac{1}{\bar{\tau}} [\mu \psi(-\mu v, t) - \psi(v, t)] \tag{51}$$

where $\mu = (M + 1)/(M - 1)$. The nth moment of Eq. (49) is

$$\frac{d}{dt} \overline{v^n(t)} = nA\overline{v^{n-1}(t)} - \frac{1}{\bar{\tau}} [1 - (-\mu)^{-n}] \overline{v^n(t)} \tag{52}$$

where the second term in Eq. (51) has been integrated by parts, assuming that $\psi v^n \to 0$ as $|v| \to \infty$. If $\bar{v} = 0$ initially, then the $n = 1$ equation yields

$$\overline{v(t)} = \frac{A\bar{\tau}}{1 + \mu^{-1}} [1 - e^{-(1 + \mu^{-1})t/\bar{\tau}}] \tag{53}$$

The nth moment is given by

$$\overline{v^n(t)} = nA \int_0^t e^{-[1 - (-\mu)^{-n}](t - t')/\bar{\tau}} \overline{v^{n-1}}(t')\, dt' \tag{54}$$

from which it is apparent that if $\overline{v^{n-1}}(t)$ possesses a limit as $t \to \infty$, then so does $\overline{v^n(t)}$. Since the average velocity, Eq. (53), saturates as $t \to \infty$, it follows that all higher moments do so as well. It is only in the limit $M \to \infty$ that $\overline{v^2}$ and higher moments increase without bound as $t \to \infty$.

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